Differential forms and smoothness of quotients by reductive groups

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Abstract

Let $\pi : X \longrightarrow Y$ be a good quotient of a smooth variety X by a reductive algebraic group G and $1 \leq k \leq \dim(Y)$ an integer. We prove that if, locally, any invariant horizontal differential k-form on X (resp. any regular differential k-form on Y) is a Kähler differential form on Y then $\operatorname{codim}(Y_{\text{sing}}) > k+1$. We also prove that the dualizing sheaf on Y is the sheaf of invariant horizontal $\dim(Y)$-forms.

Introduction

Let $\pi : X \longrightarrow Y$ be a good quotient of a smooth variety X by a reductive algebraic group G. How one can bound the dimension of the singular locus of Y? Since there exists no natural embedding of Y in some smooth variety, it seems difficult to describe the n-th Fitting ideal of the sheaf Ω^1_Y. J. Fogarty suggests a different approach to this problem by raising in [Fog88] the following questions (all schemes are assumed to be of finite type over a field of characteristic 0) :

Question *Let G be a finite group acting on a smooth variety X and $\pi : X \longrightarrow Y$ the quotient. Is the natural morphism*

$$\Omega^1_Y \longrightarrow (\Omega^1_X)^G$$

surjective if and only if Y is smooth?

In that article J. Fogarty verifies that the surjectivity condition is indeed necessary. He also proves that, when the group G is abelian, this condition is sufficient ([Fog88, Lemma 5]).

Observe that the module $(\Omega^1_X)^G$ is naturally isomorphic to $\Omega^1_Y{}^{\vee\vee}$ and, the variety Y being normal, also isomorphic to the module ω^1_Y of regular 1-forms (cf. appendix A) and to the module $i_*\Omega^1_{Y_{\text{smth}}}$ (here i denotes the inclusion $Y_{\text{smth}} \subset y$). It is also

easily checked that this problem reduces to the case where X is a rational representation of G. In particular when $G \subset \mathrm{SL}(\mathbb{C}^2)$, then $Y = \mathbb{C}^2/G$ is a complete intersection and one can give an affirmative answer to the question above. However, already in dimension 2 (i.e. $G \subset \mathrm{GL}(\mathbb{C}^2)$) this question appears to be quite tricky.

Recently M.Brion proved the following result :

Theorem ([Bri98, Theorem 1]) *Let G be a reductive algebraic group acting on a smooth affine variety X, and let $\pi : X \longrightarrow Y$ be the quotient. If Y is smooth then the natural morphism*

$$(d\pi)^G \quad : \quad \Omega_Y \longrightarrow (\Omega_{X,G})^G$$

is an isomorphism.

Here $(\Omega_{X,G})^G$ is the differential graded algebra of *invariant horizontal differential forms* and $(d\pi)^G$ is the morphism of differential graded algebras induced by the cotangent morphism $d\pi$ (see section 1). When G is finite, it is isomorphic to $(\Omega_X)^G$. This last theorem clearly suggests to reformulate and investigate Fogarty's question in the more general context of quotients by reductive groups.

The main theorem we prove in this paper is the following, thus giving a partial answer to Fogarty's question and also a strong converse to Brion's theorem :

Theorem (5.1) *Let G be a reductive algebraic group acting on a smooth affine variety X, with quotient map $\pi : X \longrightarrow Y$ and let k be an integer with $1 \leq k \leq \dim(Y)$. The morphism $(d\pi^k)^G$ is surjective in codimension $k+1$ if and only if Y is smooth in codimension $k+1$.*

We stated these results for affine G-schemes, but it is easy to see that they generalize immediately to the case of good quotients (i.e. affine uniform categorical quotient morphisms $\pi : X \longrightarrow Y$, with the terminology of [MF82]).

In the case of finite abelian groups we also prove :

Theorem (6.1) *Let G be a finite abelian group acting on a smooth affine scheme X with quotient $\pi : X \longrightarrow Y$ and let k be an integer with $1 \leq k \leq \dim(X)$. The morphism $(d\pi^k)^G$ is surjective if and only if Y is smooth.*

This improves the previous result of Fogarty and also shows that, with the hypothesis of (5.1) smoothness in codimension $k+1$ doesn't imply that $(d\pi^k)^G$ (or c_Y^k, see below) is surjective.

In order to prove these theorems it is important to understand how $(\Omega_{X,G})^G$ compares to other sheaves of differentials on Y, in particular to the sheaves $\tilde{\Omega}_Y$ and ω_Y (respectively, the sheafs of *absolutely regular* and *regular differential forms*. Cf. appendix A). In his article [Bri98], M. Brion observed that, as a corollary to his theorem and under the additional condition that *no invariant divisors is mapped by π onto a closed subscheme of codimension ≥ 2 in Y*, there are isomorphisms

$(\Omega_{X,G})^G \simeq \Omega_Y{}^{\vee\vee} \simeq \omega_Y$. This comparison problem is also closely related to the more classical problem of describing the dualizing sheaf of a quotient (by a reductive group) variety as a sheaf of invariants. It has been extensively studied by F. Knop in [Kno89], but the expression he obtains for ω_Y^n (the canonical sheaf if $n = \dim(Y)$) is again dependent on the existence of the preceding "bad divisors".

Here, using a general machinery of Kähler (resp. absolutely regular) horizontal differential forms (sections 1 and 4) we obtain the following comparison statement :

Proposition (4.4) *Let G be a reductive algebraic group, X be a smooth affine G-scheme and $\pi : X \longrightarrow Y$ the quotient. There is a sequence of inclusions :*

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq \omega_Y$$

which are equalities on the smooth locus of Y.

This together with a theorem of Boutot ([Bou87]) leads to the following simple description of the dualizing sheaf :

Corollary (4.5) *Let G be a reductive algebraic group, X be a smooth affine G-scheme with quotient map $\pi : X \longrightarrow Y$ and let $n = \dim(Y)$. Then the dualizing sheaf on Y, ω_Y^n, is isomorphic to $(\Omega^n_{X,G})^G$.*

Our Proposition 4.4 also leads to a more intrinsic version of (5.1) :

Theorem (5.2) *Let Y be the quotient of a smooth affine variety by a reductive algebraic group and let k be an integer with $1 \leq k \leq \dim(Y)$. The fundamental class morphism c_Y^k is surjective in codimension $k+1$ if and only if Y is smooth in codimension $k+1$.*

Note that this result apply in particular when Y is a variety with toroidal singularities. Indeed, it is proved in [Cox95] that any toric variety can be realized as the good quotient of an open subset of an affine space $\mathbb{A}^n$ by a torus. In fact, for quotient by tori, we expect that a statement similar to (6.1) might hold.

A smoothness criterion much like (5.2) also holds when Y is locally a complete intersection ([Vet70] or [Jam00]. Note by the way that quotient singularities which are complete intersections are "exceptionnal" and must be singular in codimension 2). Even more generally, one may conjecture that for a variety Y with reasonnable singularities (see [KW88, 5.22, p107] in appendix A) *c_Y^k is surjective in codimension k if and only if Y is smooth in codimension k* (the "$k+1$" in (5.2) is clearly a gift of the local quasi-homogeneous structure).

Finally, combined with results of H. Flenner ([Fle88], and van Sraten-Steenbrink [vSS85] in the case of isolated singularities) proposition 4.4 implies that for $0 \leq k < \operatorname{codim}(Y_{\text{sing}}) - 1$, we have $\tilde{\Omega}_Y^k \simeq (\Omega^k_{X,G})^G \simeq \omega_Y^k$. However, the following question (as far as we know) remains open : *Under the hypotheses of (4.4) do we have in general isomorphisms $\tilde{\Omega}_Y \simeq (\Omega_{X,G})^G \simeq \omega_Y$ or at least $(\Omega_{X,G})^G \simeq \omega_Y$?*

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Notation and conventions We work over a fixed field $\mathbf{k}$ of characteristic 0 with algebraic closure $\bar{\mathbf{k}}$. All the schemes we consider are of finite type over $\mathbf{k}$. For such a scheme X, we denote by Ω_X the differential graded algebra $\oplus_{k\geq 0}\Omega^k_{X/\mathbf{k}}$ of Kähler differentials, and write Ω^k_X for $\Omega^k_{X/\mathbf{k}}$.

For G an algebraic group and a G-scheme X, we denote by G-$\mathcal{O}_X$-mod the category of G-equivariant $\mathcal{O}_X$-modules.

An affine $\mathbb{G}_m$-scheme X is said to be quasi-conical (this is an ugly terminology, but, we believe it is consistent with the algebraic definitions of homogeneous and quasi-homogeneous ideals) if $\mathcal{O}_X$ is generated by homogeneous sections of non-negative weights. We recall that X is said to be conical when $\mathcal{O}_X$ is generated by homogeneous sections of weight 1.

By differential operator, we mean differential operator relative to $\mathbf{k}$ in the sense of [Gro67, 16.8].

We denote by Γ the decreasing filtration by codimension of the support : Let c be an integer. For any $\mathcal{O}_X$-module M and $U \subset X$ an open subset, $\Gamma_c M(U)$ is the subgroup of $M(U)$ consisting of the sections having support of codimension $\geq c$ in X. We write $\Gamma_{(c)}$ for Γ_c/Γ_{c+1} and $\bar{M}$ for $\Gamma_{(0)}M$. In particular, when X is integral, $\Gamma_1 M$ is the submodule of torsion elements and $\bar{M} = \Gamma_{(0)}M$ is M modulo torsion. We recall that this filtration is preserved by differential operators and in particular by $\mathcal{O}_X$-linear morphisms. These definitions extend to categories of complexes in the obvious way.

By a desingularisation of X, we always mean a desingularisation of X_{red}. We take ([EV98]) as a general reference for resolution of singularities, in particular for the existence of equivariant resolutions.

1 Horizontal differentials

Let G be an algebraic group, $\mathfrak{g}$ its Lie algebra considered as a G-module via the adjoint representation, and X a G-scheme. We will also consider G as a G-scheme by the action of G on itself by inner automorphism. We have the following diagram of equivariant maps :

$$\begin{tikzcd} G & G \times X \arrow[l, "p"'] \arrow[r, shift left, "\mu"] \arrow[d, "q"] & X \arrow[l, shift left, "s"] \\ & X & \end{tikzcd}$$

where p and q are the projections, μ is the action map and s is the section of μ defined by $x \mapsto (e, x)$. This induces the following diagram of G-equivariant coherent

modules on $G \times X$:

$$\begin{tikzcd} \mu^*\Omega^1_X \arrow[r, "d\mu"] \arrow[dr] & \Omega^1_{G\times X} = p^*\Omega^1_G \oplus q^*\Omega^1_X \arrow[d] \\ & p^*\Omega^1_G \end{tikzcd}$$

Taking the pull-back by s of the diagonal morphism above, we obtain a morphism

$$d\mu^1_{X,G} \quad : \quad \Omega^1_X \longrightarrow s^*p^*\Omega^1_G = \mathfrak{g}^\vee \otimes \mathcal{O}_X$$

We then define a morphism $d\mu_{X,G} : \Omega_X \longrightarrow \Omega_X \otimes \mathfrak{g}^\vee$ as follows

$$\begin{aligned} d\mu^k_{X,G} \quad &: \quad \Omega^k_X \longrightarrow \Omega^{k-1}_X \otimes \mathfrak{g}^\vee \\ d\mu^k_{X,G}(df_1 \wedge \ldots \wedge df_k) \quad &= \quad \sum_{i=1}^{k} (-1)^{k-i} df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k \otimes d\mu^1_{X,G}(df_i), \end{aligned}$$

1.1 Remark For an alternative and more rigourous construction of the morphisms above, using 'multilinear homological algebra', we refer to [Jam00].

Definition 1.2 *The G-equivariant module $\Omega^k_{X,G} = \mathrm{Ker}(d\mu^k_{X,G})$ is called the module of horizontal k-forms. We denote by $\Omega_{X,G}$ the graded algebra $\oplus_{k\geq 0}\Omega^k_{X,G}$.*

The sections of $\Omega_{X,G}$ consists of those forms whose interior product with any vector field induced by the group action vanishes.

The preceeding construction is natural in X. Thus, for any equivariant map $f : X \longrightarrow Y$ the cotangent morphism induces morphisms $f^*\Omega^k_{Y,G} \longrightarrow \Omega^k_{X,G}$. It is also clear from the construction that if the action of G is trivial then $d\mu^1_{X,G} = 0$ and consequently we have $\Omega^k_{X,G} = \Omega^k_X$. From these remarks, we deduce :

Proposition 1.3 *Let $\pi : X \longrightarrow Y$ be a G-invariant morphism, then the cotangent morphism $d\pi : \pi^*\Omega_Y \longrightarrow \Omega_X$ factors through $\Omega_{X,G} \subset \Omega_X$.*

1.4 Remark This last proposition applies in particular when π is a categorical quotient of X. Assume that X is affine and that G is a reductive linear group. Let $\pi : X \longrightarrow Y$ be the quotient of X. By (1.3) there is a morphism $\pi^*\Omega_Y \longrightarrow \Omega_{X,G}$ and therefore a morphism

$$(d\pi)^G \quad : \quad \Omega_Y \longrightarrow (\Omega_{X,G})^G$$

of coherent modules on Y. Under the additional assumption that X is smooth, then $(\Omega_{X,G})^G$ is a torsion-free module and by ([Bri98, Theorem 1]) the morphism $(d\pi)^G$ is generically an isomorphism. Consequently, the kernel of $(d\pi)^G$ is exactly the torsion of Ω_Y and we have an inclusion : $\bar{\Omega}_Y \subseteq (\Omega_{X,G})^G$.

We now give some elementary properties of this construction :

Lemma 1.5 *Let $f : X \longrightarrow Y$ be an equivariant map of G-schemes. Assume that the adjoint morphism $\Omega_Y \longrightarrow f_*\Omega_X$ is injective. Then the diagram :*

$$\begin{tikzcd}
\Omega_{Y,G} \arrow[r] \arrow[d] & \Omega_Y \arrow[d] \\
f_*\Omega_{X,G} \arrow[r] & f_*\Omega_X
\end{tikzcd}$$

is a fiber product diagram where all the morphisms are injective.

In other words, under the assumption, a differential form is horizontal if and only if its pull-back is.

Proof of 1.5 The statement is an easy consequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{Y,G} \arrow[r] \arrow[d] & \Omega_Y \arrow[r, "d\mu_{Y,G}"] \arrow[d] & \Omega_Y \otimes \mathfrak{g}^\vee \arrow[d] \\
0 \arrow[r] & f_*\Omega_{X,G} \arrow[r] & f_*\Omega_X \arrow[r, "f_*d\mu_{X,G}"] & f_*\Omega_X \otimes \mathfrak{g}^\vee
\end{tikzcd}$$

where the two vertical morphisms on the left are injective by assumption. ■

Lemma 1.6 *Let G be an algebraic group and $f : X \longrightarrow Y$ be a principal G-fibration. Then the natural morphism $df : f^*\Omega_Y \longrightarrow \Omega_{X,G}$ is an isomorphism.*

One is reduced to proving the statement in the case of a trivial G-fibration where this is obvious.

Lemma 1.7 *Let G and H be algebraic groups acting on a scheme X. The natural commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{X,G} \otimes \mathfrak{h}^\vee \arrow[r] & \Omega_X \otimes \mathfrak{h}^\vee \arrow[r, "d\mu_{X,G} \otimes \mathfrak{h}^\vee"] & \Omega_X \otimes \mathfrak{g}^\vee \otimes \mathfrak{h}^\vee \\
0 \arrow[r] & \Omega_{X,G} \arrow[r] \arrow[u] & \Omega_X \arrow[r, "d\mu_{X,G}"] \arrow[u, "d\mu_{X,H}"] & \Omega_X \otimes \mathfrak{g}^\vee \arrow[u, "-d\mu_{X,H} \otimes \mathfrak{g}^\vee"'] \\
0 \arrow[r] & \Omega_{X,G} \cap \Omega_{X,H} \arrow[r] \arrow[u] & \Omega_{X,H} \arrow[r] \arrow[u] & \Omega_{X,H} \otimes \mathfrak{g}^\vee \arrow[u] \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

has exact rows and columns. Moreover, it induces an exact sequence :

$$0 \longrightarrow \Omega_{X,G} \cap \Omega_{X,H} \longrightarrow \Omega_X \longrightarrow \Omega_X \otimes (\mathfrak{g}^\vee \oplus \mathfrak{h}^\vee).$$

Observe, that we did not assume that the actions of G and H on X commute, therefore this diagram is only separately G and H-equivariant, but, in general, not $G \times H$-equivariant.

2 The Euler derivation

We go on using the notations of section 1. Let $T = \mathbb{G}_m = \mathrm{Spec}(\mathbf{k}[\lambda, \lambda^{-1}])$ be a one-dimensional torus with Lie algebra $\mathfrak{t}$ and X an affine T-scheme. We recall that since T is abelian, the adjoint representation is trivial, i.e. $\mathfrak{t}$ is a trivial T-module. We fix once for all an isomorphism $\mathbf{k} \simeq \mathfrak{t}$ via the left-invariant derivation $\lambda\frac{\partial}{\partial\lambda}$. Composing the dual of this last isomorphism with $d\mu^1_{X,T}$ we obtain a derivation on X :

$$\mathrm{e}_{X,T} \quad : \quad \Omega^1_X \longrightarrow \mathcal{O}_X$$

called the Euler derivation. Since X is affine, we have $X = \mathrm{Spec}(A)$ with A a graded ring. The grading of A corresponds to the weight for the T-action : A section f of $\mathcal{O}_X$ is said to be homogeneous of weight w if $\mu^* f = \lambda^w q^* f$. If f is homogeneous of weight w, we set $|f| = w$.

Proposition 2.1 *Let f be an homogeneous section of $\mathcal{O}_X$. Then :*

$$\mathrm{e}(df) = |f|f.$$

Proof of 2.1 Let $w = |f|$. We have :

$$\begin{aligned}
\mathrm{e}(df) &= \lambda\frac{\partial}{\partial\lambda} d\mu^1_{X,T}(df) \\
&= \lambda\frac{\partial}{\partial\lambda} s^*(w\lambda^{w-1} f.d\lambda) \\
&= \lambda\frac{\partial}{\partial\lambda} s^*(w\lambda^{w} f.\frac{d\lambda}{\lambda}) \\
&= \lambda\frac{\partial}{\partial\lambda} (wf.\frac{d\lambda}{\lambda}) \\
&= wf
\end{aligned}$$

as expected. ∎

Proposition 2.2 *The Euler derivation constructed above can be extended to a degree -1 endomorphism of the graded module Ω_X by setting :*

$$\begin{aligned} \mathrm{e} \quad &: \quad \Omega_X^k \longrightarrow \Omega_X^{k-1} \\ \mathrm{e}(df_1 \wedge \ldots \wedge df_k) \quad &= \quad \sum_{i=1}^{k} (-1)^{k-i} \mathrm{e}(df_i) df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_k. \end{aligned}$$

It satisfies the following two properties :

(i) $\mathrm{e}^2 = 0$.

(ii) For any two forms α, β of respective degree k and l, we have

$$\mathrm{e}(\alpha \wedge \beta) = (-1)^l \mathrm{e}(\alpha) \wedge \beta + \alpha \wedge \mathrm{e}(\beta).$$

Proof of 2.2 By direct computation. ■

We thus have constructed a complex that we will denote by (Ω_X, e).

The exterior differential algebra (Ω_X, d) is also graded by weight : A section α of (Ω_X, d) is homogeneous of weight w if $\mu^*\alpha = \lambda^w q^*\alpha$. The following properties are then easy to check :

Proposition 2.3 *Let α and β be homogeneous sections of Ω_X.*

(i) The forms $d\alpha$ and $\mathrm{e}(\alpha)$ are homogeneous and $|d\alpha| = |\mathrm{e}(\alpha)| = |\alpha|$.

(ii) The form $\alpha \wedge \beta$ is homogeneous and $|\alpha \wedge \beta| = |\alpha| + |\beta|$.

(iii) The algebra Ω_X is generated by the differentials of homogeneous sections of $\mathcal{O}_X$.

(iv) $\mathrm{Ker}(\mathrm{e}) = \Omega_{X,T}$.

Proposition 2.4 *For any homogeneous k-forms α, we have :*

$$[\mathrm{e}, d]\alpha = (-1)^k |\alpha| \alpha.$$

Proof of 2.4 This is a direct computation again. ■

Let $c \geq 0$. The operators e and d preserve the filtration by codimension of the support and therefore they induce operators on $\Gamma_c\Omega_X$ and $\Gamma_{(c)}\Omega_X$ that we again denote by e and d. Moreover, since $\Gamma_c\Omega_X$ and $\Gamma_{(c)}\Omega_X$ are also T-equivariant, the statement above remains true for these modules.

Proposition 2.5 *The submodule $(\Omega_{X,T})^T \subseteq (\Omega_X)^T$ is stable by the exterior derivative of Ω_X.*

Proof of 2.5 Keeping in mind that T-invariants are precisely homogeneous sections of null weight, the result is a direct consequence of (2.4) and (2.3 (iv)). ■

3 Horizontal differentials : Poincaré lemmas

Proposition 3.1 *Let G be a reductive algebraic group and X an affine G-scheme. Then the submodule $(\Omega_{X,G})^G \subset (\Omega_X)^G$ is stable by the exterior derivative of Ω_X.*

3.2 Remark This statement holds more generally for G a linear algebraic group. But its proof would require an algebraic construction of the Lie derivative that we did not explain here. The proof would run as follows : For $v \in \mathfrak{g}$, denotes by L_v the Lie derivative and by $< v, \cdot >$ the interior product. Then, for any section α of Ω_X we have the relation :

$$\mathrm{L}_v\alpha = d < v, \alpha > + < v, d\alpha > .$$

The statement therefore follows from the observation that L_v vanishes on $(\Omega_X)^G$.

Proof of 3.1 We recall $(\Omega_X)^G$ is obviously stable by exterior differentiation. Since G is reductive, on can find one-dimensional subtori $T_1, \ldots, T_d$ of G such that $\mathfrak{g} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_d$. Then, by (1.7), we have :

$$\Omega_{X,G} \quad = \quad \Omega_{X,T_1} \cap \ldots \cap \Omega_{X,T_d}.$$

And therefore

$$\begin{aligned}(\Omega_{X,G})^G &= (\Omega_X)^G \cap \Omega_{X,T_1} \cap \ldots \cap \Omega_{X,T_d} \\ &= (\Omega_X)^G \cap (\Omega_{X,T_1})^{T_1} \cap \ldots \cap (\Omega_{X,T_d})^{T_d}.\end{aligned}$$

By (2.5), all the terms in the intersection above are stable by d, so we can conclude that $(\Omega_{X,G})^G$ is stable by d too. ∎

Proposition 3.3 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme. Then $\Omega_{X,G}$ is stable by $\mathrm{e} = \mathrm{e}_{X,T}$. We write $(\Omega_{X,G}, \mathrm{e})$ for this subcomplex of (Ω_X, e).*

Proof of 3.3 By a direct calculation, using the explicit definitions of $d\mu_{X,G}$ and e. ∎

Therefore, if $c \geq 0$ is an integer, $\Gamma_c\Omega_{X,G}$ is also stable by e and therefore there is an induced endomorphism on $\Gamma_{(c)}\Omega_{X,G}$.

Corollary 3.4 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme. Let α be a homogeneous section (with respect to the T-action) of $(\Omega^k_{X,G})^G$. Then*

$$[\mathrm{e}, d]\alpha = (-1)^k |\alpha| \alpha.$$

Clearly, we again have a similar statement for $\Gamma_c(\Omega_{X,G})^G$, $\Gamma_{(c)}(\Omega_{X,G})^G$, $(\Gamma_c\Omega_{X,G})^G$ or $(\Gamma_{(c)}\Omega_{X,G})^G$.

Proposition 3.5 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme. Then*

$$\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big) = \mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)^T .$$

Let $c \geq 0$. Then the same relation holds for $\Gamma_c(\Omega_{X,G})^G$, $\Gamma_{(c)}(\Omega_{X,G})^G$, $(\Gamma_c\Omega_{X,G})^G$ and $(\Gamma_{(c)}\Omega_{X,G})^G$.

Proof of 3.5 Let α be a homogeneous section of $(\Omega^k_{X,G})^G \cap \mathrm{Ker}(\mathrm{e})$. Then by (3.4) we have $\mathrm{e}d\alpha = (-1)^k|\alpha|\alpha$. Therefore if $|\alpha| \neq 0$ the class of α in $\mathrm{H}_k\Big((\Omega_{X,G})^G, \mathrm{e}\Big)$ vanishes. Since $\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)^T$ is a direct factor of $\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)$, the equality is proved. ∎

Lemma 3.6 *Let X be a quasi-conical affine T-scheme. Then the pull-back morphism for the quotient map $X \longrightarrow X/\!\!/T$ induces isomorphisms :*

$$\Omega_{X/\!\!/T} \xrightarrow{\sim} (\Omega_{X,T})^T \xrightarrow{\sim} (\Omega_X)^T \subset \Omega_X.$$

Proof of 3.6 Easy, by arguments on weights. ∎

Proposition 3.7 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme, quasi-conical with respect to the T-action. Then the natural morphism*

$$\Omega_{X/\!\!/T,G} \longrightarrow (\Omega_{X,G})^T$$

induced by the G-equivariant map $X \longrightarrow X/\!\!/T$, is an isomorphism.

Proof of 3.7 By (3.6) the hypotheses of (1.5) are satisfied for the map $X \longrightarrow X/\!\!/T$. Taking T-invariants in the diagram of (1.5) together with the isomorphism $\Omega_{X/\!\!/T} \xrightarrow{\sim} (\Omega_X)^T$ gives the result. ∎

Proposition 3.8 *Let G be a reductive algebraic group and let X be an affine $G\times T$-scheme, quasi-conical with respect to the T-action. Let $d \geq c \geq 0$. There are*

isomorphisms of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Gamma_d\Omega_{X,G})^T & \longrightarrow & (\Gamma_c\Omega_{X,G})^T & \longrightarrow & (\Gamma_c/\Gamma_d\,\Omega_{X,G})^T & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{H}\Big((\Gamma_d\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c/\Gamma_d\,\Omega_{X,G})^T, \mathrm{e}\Big) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Gamma_d\Omega_{X,G})^{G\times T} & \longrightarrow & (\Gamma_c\Omega_{X,G})^{G\times T} & \longrightarrow & (\Gamma_c/\Gamma_d\,\Omega_{X,G})^{G\times T} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{H}\Big((\Gamma_d\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & \mathrm{H}\Big((\Gamma_c/\Gamma_d\,\Omega_{X,G})^G, \mathrm{e}\Big) & \longrightarrow & 0
\end{array}$$

Proof of 3.8 By (3.7) we have $(\Gamma_c\Omega_{X,G})^T \subset \Omega_{X/\!\!/T}$. Therefore e vanishes for all the complexes involved in the first isomorphism and this proves the first statement. For the second one, take G-invariants in the first diagram and use (3.5). ■

One might understand the next two statements as a natural generalisation, with e and d exchanged, of the Poincaré Lemma to singular varieties with reductive group action :

Corollary 3.9 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme, quasi-conical with respect to the T-action. Then the G-equivariant map $X \longrightarrow X/\!\!/T$ induces an isomorphism*

$$(\Omega_{X/\!\!/T,G})^G \xrightarrow{\sim} \mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big)\,.$$

Corollary 3.10 *Let G be a reductive algebraic group and let X be an affine $G \times T$-scheme, quasi-conical with respect to the T-action and such that $X/\!\!/T = \mathrm{Spec}(\mathbf{k})$. Then*

$$\mathrm{H}\Big((\Omega_{X,G})^G, \mathrm{e}\Big) = \mathrm{H}\Big((\bar{\Omega}_{X,G})^G, \mathrm{e}\Big) = \mathbf{k}.$$

In particular, in the case of a trivial action of G on a variety and under the preceding hypotheses we have exact complexes

$$\cdots \longrightarrow \Omega^n_X \longrightarrow \cdots \longrightarrow \Omega^1_X \longrightarrow \mathcal{O}_X \longrightarrow \mathbf{k} \longrightarrow 0$$

$$0 \longrightarrow \bar{\Omega}^n_X \longrightarrow \cdots \longrightarrow \bar{\Omega}^1_X \longrightarrow \mathcal{O}_X \longrightarrow \mathbf{k} \longrightarrow 0$$

4 Absolutely regular horizontal differentials

In this section, we merge the construction of horizontal differentials and the content of appendix A.2.

Let X be a G-scheme and $f : \tilde{X} \longrightarrow X$ a G-equivariant desingularisation. We denote by $\tilde{\Omega}_{X,G}$ the sheaf $f_*\Omega_{\tilde{X},G}$. This definition is independent of the choice of f, as in the non-equivariant case, since two equivariant resolutions of singularities can be covered by a third one.

By construction, we have natural equivariant morphisms

$$\Omega_{X,G} \longrightarrow \tilde{\Omega}_{X,G} \longrightarrow i_*\Omega_{X_{\mathrm{smth}},G}$$

where i is the inclusion $X_{\mathrm{smth}} \subset X$. Therefore, when X is reduced, we have :

$$\Omega_{X,G} \longrightarrow \bar{\Omega}_{X,G} \subset \tilde{\Omega}_{X,G} \subset i_*\Omega_{X_{\mathrm{smth}},G}.$$

Proposition 4.1 *Let $f : X \longrightarrow Y$ be an equivariant dominant morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \Omega_{X,G} \arrow[r] & \tilde{\Omega}_{X,G} \\ f^*\Omega_{Y,G} \arrow[r] \arrow[u] & f^*\tilde{\Omega}_{Y,G} \arrow[u] \end{tikzcd}$$

Proposition 4.2 *Let $f : X \longrightarrow Y$ be an invariant dominant morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \Omega_{X,G} \arrow[r] & \tilde{\Omega}_{X,G} \\ f^*\Omega_{Y} \arrow[r] \arrow[u] & f^*\tilde{\Omega}_{Y} \arrow[u] \end{tikzcd}$$

Proposition 4.3 *Let $f : X \longrightarrow Y$ be a proper equivariant birational morphism. Then the morphism $\tilde{\Omega}_{Y,G} \longrightarrow f_*\tilde{\Omega}_{X,G}$ is an isomorphism.*

With this at hand, we can give a partial answer to the question raised by M. Brion ([Bri98, after Theorem 2]) :

Proposition 4.4 *Let G be a reductive algebraic group, X be a smooth affine G-scheme and $\pi : X \longrightarrow Y$ the quotient. There is a sequence of inclusions :*

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq \omega_Y$$

which are equalities on the smooth locus of Y.

Corollary 4.5 *Let G be a reductive algebraic group, X be a smooth affine G-scheme with quotient map $\pi : X \longrightarrow Y$ and let $n = \dim(Y)$. Then the dualizing sheaf on Y, ω_Y^n, is isomorphic to $(\Omega^n_{X,G})^G$.*

Proof of 4.4 Since $\Omega_{X,G} = \tilde{\Omega}_{X,G}$, by (4.2) we have inclusions $\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G$ of torsion-free modules. Moreover, by the theorem of Brion ([Bri98, Theorem 1]), these are isomorphisms outside the closed subset Y_{sing}, therefore outside a closed subset of codimension ≥ 2. Thus the modules involved have isomorphic biduals and we obtain :

$$\bar{\Omega}_Y \subseteq \tilde{\Omega}_Y \subseteq (\Omega_{X,G})^G \subseteq {\Omega_Y}^{\vee\vee} = \omega_Y.$$

■

Proof of 4.5 It is then a direct consequence of the fact that Y has rational singularities ([Bou87]). Indeed, this implies that $\tilde{\Omega}^n_Y \xrightarrow{\sim} \omega^n_Y$. ■

4.6 Remark If one assume that all the points of X are strongly stable for the action of G, i.e., that for all closed points $x \in X$, the orbit Gx is closed and the stabilizer G_x is finite, then there are isomorphisms

$$\tilde{\Omega}_Y \xrightarrow{\sim} (\Omega_{X,G})^G \xrightarrow{\sim} \omega_Y.$$

To prove this, one can assume that the group G is already finite (use the Etale Slice Theorem as in the last reduction step in (5) below). With this assumption made it is easily seen that $\Omega_{X,G} = \Omega_X$ (here $\mathfrak{g} = (0)$) and that consequently $(\Omega_X)^G = \omega_Y$. It therefore remains to see that $\tilde{\Omega}_Y = (\Omega_X)^G$. This can be done as follows.

We have a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{\pi}"] \arrow[d, "g"] & \tilde{Y} \arrow[d, "f"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

where f is a resolution of singularities for Y and $\tilde{X}$ is the normalization of the component birational to X in $X \times_Y \tilde{Y}$. The group G acts naturally on $\tilde{X}$ and the map $\tilde{\pi}$ is the quotient morphism. We thus have a morphism

$$\Omega_{\tilde{Y}} \longrightarrow (\tilde{\pi}_* \tilde{\Omega}_{\tilde{X}})^G$$

induced by $\tilde{\pi}$. Since $\tilde{X}$ is normal it is an isomorphism in codimension 1 and since $\Omega_{\tilde{Y}}$ is locally free it is in fact an isomorphism (recalling that $\tilde{\Omega}_{\tilde{X}}$ is torsion-free). Consequently, we have

$$\tilde{\Omega}_Y = f_* \Omega_{\tilde{Y}} = f_*(\tilde{\pi}_* \tilde{\Omega}_{\tilde{X}})^G = (\pi_* g_* \tilde{\Omega}_{\tilde{X}})^G = (\pi_* \Omega_X)^G.$$

This proves our claim.

5 Invariant horizontal differentials and smoothness

In this section we give proofs for the results stated in the introduction :

Theorem 5.1 *Let G be a reductive algebraic group acting on a smooth affine variety X, with quotient map $\pi : X \longrightarrow Y$ and let k be an integer with $1 \leq k \leq \dim(Y)$. The morphism $(d\pi^k)^G$ is surjective in codimension $k+1$ if and only if Y is smooth in codimension $k+1$.*

Theorem 5.2 *Let Y be the quotient of a smooth affine variety by a reductive algebraic group and let k be an integer with $1 \leq k \leq \dim(Y)$. The fundamental class morphism c_Y^k is surjective in codimension $k+1$ if and only if Y is smooth in codimension $k+1$.*

Proof of 5.1 After deleting a closed subset of codimension $> k+1$ we may assume that the morphism $(d\pi)^G : \Omega_Y \longrightarrow (\Omega_{X,G})^G$ is surjective in degree k, i.e. that we have a surjection $\Omega_Y^k \longrightarrow (\Omega_{X,G}^k)^G$ and we want to prove that under this hypothesis the singular locus of Y has codimension $> k+1$.

The proof, now divides in five steps.

Etale slices

Quite generally, let $H \longrightarrow G$ be a map of reductive algebraic groups and W an affine H-scheme together with an H-equivariant map $j : W \longrightarrow X$. We let $G \times H$ act on $G \times W$ in the following way : $(g,h)(g',w) = (gg'h^{-1}, hw)$ and denote by $f : G \times W \longrightarrow G \times_H W$ the quotient by $1 \times H$. Observe that since $1 \times H$ acts freely on $G \times W$, the map f is a principal fibration and therefore is smooth. We obtain commutative diagram of $G \times H$-schemes :

$$\begin{array}{ccccc} G \times W & \xrightarrow{f} & G \times_H W & \xrightarrow{\bar{\mu}(G \times_H j)} & X \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ W & \longrightarrow & W/\!\!/H & \longrightarrow & X/\!\!/G \end{array} \qquad (1)$$

where the vertical maps are quotients by G, the horizontal maps in the left-square are quotients by $1 \times H$ and $\bar{\mu}$ is the factorization of the $1 \times H$-invariant map μ ($1 \times H$ acts trivially on X).

For $y \in Y$ a closed point, we denote by $T_y \subset X_y$ the unique closed orbit over y. Let $x \in T_y$ be a closed point with (necessarily) reductive stabilizer $H = G_x$. The Etale Slice theorem of Luna ([Lun73, pp 96–99]), asserts the following : There

exists a smooth locally closed, H-stable subvariety W of X such that $x \in W$, $G.W$ is an open set and such that in the natural commutative diagram (1) the right-square is cartesian with etale horizontal maps (i.e. an etale base change diagram). Moreover, letting $N = N_{T_y/X}(x)$ be the normal space at x of the orbit T_y, understood geometrically as a rational representation of H, there is a natural map of H-schemes $\rho : W \longrightarrow N$, etale at 0, which induces a commutative diagram :

$$\begin{tikzcd} G \times_H N \arrow[d, "\phi"] & G \times_H W \arrow[l, "G\times_H \rho"'] \arrow[r, "\bar{\mu}(G\times_H j)"] \arrow[d] & X \arrow[d, "\pi"] \\ N/\!\!/H & W/\!\!/H \arrow[l] \arrow[r] & X/\!\!/G \end{tikzcd} \tag{2}$$

where the two squares are cartesian and the horizontal maps are etale neighbourhoods.

Stratification by slice type

We again refer to ([Lun73, pp 100–102]). Let $H \subseteq G$ be a reductive subgroup and N an H-module. We have a commutative diagram :

$$\begin{tikzcd} G \times N \arrow[r, "f"] \arrow[dr] & G \times_H N \arrow[d] \\ & G/H \end{tikzcd} \tag{3}$$

which realizes $G \times_H N$ as the total space of a G-equivariant vector bundle over the affine homogeneous space G/H with fiber at 1 equals to N. Conversely let N be a G-equivariant vector bundle over an affine G-homogeneous base T. Let $t \in T$ be a closed point then $N(t)$ is a G_t-module and G_t is reductive. Thus we have an equivalence between the set $\{(H, N)\}$ up to conjugacy and the isomorphism classes of G-equivariant vector bundles over affine homogeneous bases. We denote by $\mathcal{M}(G)$ any of those sets and classes by brackets $[\,]$.

By the preceding, we thus have a map $\mu : Y(\bar{\mathbf{k}}) \longrightarrow \mathcal{M}(G)$ which sends y to the isomorphism class $[N_{T_y/X} \longrightarrow T_y]$ or equivalently to the "conjugacy class" $[H, N]$ with the notations of the preceding section. Let $\nu \in \mathcal{M}(G)$, then the set $\mu^{-1}(\nu)$ is a locally closed subset of Y, smooth with its reduced scheme structure. We will denote by Y_ν this smooth locally closed subscheme of Y. Moreover the collection $\{Y_\nu\}_{\nu \in \mathcal{M}(G)}$ is a finite stratification of Y (in particular μ has finite image). Therefore, the map μ can be extended to all the points of Y : Let $Z \subset Y$ be an irreducible closed subset, then there exists a unique $\nu \in \mathcal{M}(G)$ such that $Z \cap Y_\nu$ is dense in Z

and one can set $\mu(Z) = \nu$. Observe that $\mu(Z)$ is the slice type of a general point of Z.

Another important fact about μ is that it is compatible with strongly etale (also called excellent) morphisms : Given such a map φ between smooth affine G-schemes, we have $\mu(\varphi /\!\!/ G) = \mu$.

We now look closer to G-schemes of the kind $G \times_H N$ and their quotients by G. Write N_H for the canonical complementary submodule to N^H in N : $N = N^H \times N_H$. Then in the construction of $G \times_H N$, N^H is a trivial H-module and therefore the diagram obtained when W is replaced by N in the left square of (1) reads :

$$\begin{tikzcd} N^H \times (G \times N_H) \arrow[r, "f"] \arrow[d, "p"] & N^H \times (G \times_H N_H) \arrow[d, "\phi"] \\ N^H \times N_H \arrow[r, "\psi"] & N^H \times (N_H /\!\!/ H) \end{tikzcd} \tag{4}$$

Let $\nu \in \mathcal{M}(G)$ be the class of (H, N), then $((G \times_H N) /\!\!/ G)_\nu = N^H \times 0 \subseteq N /\!\!/ H$. One can convince oneself of this fact through the description of $G \times_H N$ as an equivariant vector bundle over G/H.

Reduction to an isolated singularity

First, it is harmless to assume that the singular locus of Y, Y_{sing} is irreducible. Let $\mu(Y_{\text{sing}}) = \nu = [H, N]$ and let $y \in Y_{\text{sing}} \cap Y_\nu$ be a general closed point. By standard etale base change arguments in the diagram (2), our hypothesis and our conclusion hold for π at y if and only if they respectively hold for ϕ at 0. We can therefore assume that $X = G \times_H N$, $\pi = \phi$ and $Y = N /\!\!/ H$.

Now, with the notations of (4), it is clear that $Y_{\text{sing}} = N^H \times (N_H /\!\!/ H)_{\text{sing}}$. On the other hand $Y_\nu = N^H \times 0$ and, since $\mu(Y_{\text{sing}}) = \nu$, the closed subset Y_ν should cut a dense open set on Y_{sing}. Consequently, we must have $Y_\nu = Y_{\text{sing}}$ and thus $(N_H /\!\!/ H)_{\text{sing}} = 0$.

Let $\pi_H : X_H = G \times_H N_H \longrightarrow Y_H = N_H /\!\!/ H$ be the quotient map by G, then clearly $\pi = N^H \times \pi_H$. Let k be an integer, then the map $(d\pi)^G$ is diagonal with respect to the decompositions :

$$\begin{aligned} \left(\Omega^k_{X,G}\right)^G &= \bigoplus_{i=0}^{k} \Omega^i_{N^H} \boxtimes \left(\Omega^{k-i}_{X_H,G}\right)^G \\ \Omega^k_Y &= \bigoplus_{i=0}^{k} \Omega^i_{N^H} \boxtimes \Omega^{k-i}_{Y_H} \end{aligned}$$

Therefore $(d\pi)^G$ is surjective in degree k if and only if $(d\pi_H)^G$ is surjective in all degrees $k - \dim\left(N^H\right), \ldots, k$.

To conclude, we can therefore make the extra assumption that $Y = X/\!\!/G = N/\!\!/H$ has only an isolated singularity at 0. And one should notice that the theorem remains in fact only to be proved when $k = \dim(Y) - 1$ or $\dim(Y)$, since, otherwise ($k < \dim(Y) - 1$) the statement is obviously true.

Reduction to the case of a representation

We keep in mind all the identifications and assumptions made previously. Recalling diagram (4) and applying lemmas (1.6) and (1.7) to the fibration f, we have an exact sequence

$$0 \longrightarrow f^*\Omega_{G\times_H N,G} \longrightarrow \Omega_{G\times N,G} \longrightarrow \Omega_{G\times N,G} \otimes \mathfrak{h}^\vee.$$

Taking G-invariants together with lemma (1.6) for p leads to the exact sequence :

$$0 \longrightarrow \left(f^*\Omega_{G\times_H N,G}\right)^G \longrightarrow \Omega_N \longrightarrow \Omega_N \otimes \mathfrak{h}^\vee$$

Therefore, we have proved that $\left(f^*\Omega_{G\times_H N,G}\right)^G = \Omega_{N,H}$. Taking H-invariants, we obtain

$$\left(\Omega_{N,H}\right)^H = \left(f^*\Omega_{G\times_H N,G}\right)^{G\times H} = \left(\Omega_{G\times_H N,G}\right)^G.$$

One can then conclude, that the hypothesis and the conclusion of the theorem hold for ϕ if and only if they respectively hold for ψ. Thus we are reduced to prove the theorem in the case where X is a rational representation of G with $X/\!\!/G$ having only an isolated singularity at the origin.

Conclusion

Carrying on, X is now a rational G-module with quotient $\pi : X \longrightarrow Y$, such that Y has only an isolated singularity at the origin. We recall the hypothesis in the theorem : The morphism $(d\pi)^G$ is surjective in degree $k \leq \dim(Y)$. We must prove that Y is smooth in codimension $k+1$. Thus we have to prove that if $k = \dim(Y)$ or $\dim(Y) - 1$ then Y is smooth.

The one dimensional torus $T = \mathbb{G}_m$ acts on X by homothety and this action commutes with the action of G. Thus X is a $G \times T$ scheme and Y is a T-scheme. Both X and Y are quasi-conical and $X/\!\!/T = Y/\!\!/T = \mathrm{Spec}(\mathbf{k})$.

Let $n = \dim(Y)$. Applying (3.10) to X and Y we obtain an injective morphism of exact complexes (the kernel of $(d\pi)^G$ is exactly the torsion of Ω_Y, cf. remark 1.4) :

$$\begin{array}{ccccccccccccccc}
\left(\Omega_{X,G}\right)^G & & 0 & \rightarrow & \left(\Omega^n_{X,G}\right)^G & \rightarrow & \left(\Omega^{n-1}_{X,G}\right)^G & \rightarrow & \cdots & \rightarrow & \left(\Omega^1_{X,G}\right)^G & \rightarrow & \mathcal{O}_Y & \rightarrow & \mathbf{k} \rightarrow 0 \\
{\scriptstyle (d\pi)^G}\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \| & & \| \\
\bar{\Omega}_Y & & 0 & \longrightarrow & \bar{\Omega}^n_Y & \longrightarrow & \bar{\Omega}^{n-1}_Y & \longrightarrow & \cdots & \longrightarrow & \bar{\Omega}^1_Y & \longrightarrow & \mathcal{O}_Y & \rightarrow & \mathbf{k} \rightarrow 0
\end{array}$$

From this diagram, we deduce that if $(d\pi)^G$ is surjective in degree $n-1$, then it is also surjective in degree n. Therefore we have an isomorphism $\bar{\Omega}^n_Y \xrightarrow{\sim} (\Omega^n_{X,G})^G$. Moreover, by proposition (4.4) we know that $(\Omega^n_{X,G})^G = \omega^n_Y = \Omega^{n\,\vee\vee}_Y$. Thus $\bar{\Omega}^n_Y$ is a reflexive module.

Recall that by the theorem of Boutot ([Bou87]), Y has rational singularities and in particular is normal and Cohen-Macaulay and that ω^n_Y is then the dualizing module of Y. The fundamental class map c ([KW88, 5.2 p 91, 5.15 p 99], [EZ78] and appendix A), in degree n, factors through :

$$\begin{tikzcd} \Omega^n_Y \arrow[r, "c"] \arrow[d] & \omega^n_Y \\ \bar{\Omega}^n_Y \arrow[ur] & \end{tikzcd}$$

But $\bar{\Omega}^n_Y$ is reflexive and, since Y is normal, c is an isomorphism in codimension 1. Therefore c is necessarily surjective. We now invoke a theorem of Kunz and Waldi ([KW88, 5.22 p 107]) to conclude that Y is smooth.

The proof of theorem 5.1 is complete. ■

Proof of 5.2 Using (5.1), we can a give a straightforward proof of the result : By (4.4) the hypotheses of (5.1) are satisfied for the same integer k. ■

6 The case of abelian finite groups

Let G be a finite group acting on a quasi-projective scheme X and let $\pi : X \longrightarrow Y$ be the quotient.

For an element $g \in G$, we denote the closed subscheme of g-fixed points by X^g and for a point $x \in X$, we denote its stabilizer (also called isotropy subgroup) by G_x. We then define a increasing filtration of G by normal subgroups in the following way : For $k \geq 0$ an integer we set $G^k =< g \in G, \forall x \in X^g, \operatorname{codim}(X^g, x) \leq k >$. In particular G^1 is the subgroup generated by the *pseudo-reflections* in G. For a point $x \in X^g$, if $\operatorname{codim}(X^g, x) \leq 1$ then g is said to be a *pseudo-reflection at x*. When X is smooth, this condition is satisfied if and only if locally at x, the diagonal form of g is of the kind $(\zeta, 1, \ldots, 1)$ for some root of unity ζ. Clearly g is a pseudo-reflection if and only if it is a pseudo-reflection at all the points of X^g.

When $G^1 = (1)$ one says that G is a *small* group of automorphisms of X. In this case, by standard ramification theory, the quotient map is unramified in codimension one. When $G = G^1$ one says that G is generated by pseudo-reflections. We now recall the classical

Theorem (Shephard-Todd, Chevalley, Serre [ST54, Ser68]) *With the preceding notations, the following conditions are equivalent :*

(i) The quotient Y is smooth.

(ii) *For all $x \in X$, the group G_x is generated by the pseudo-reflections at x.*

(iii) *The $\mathcal{O}_Y$-module $\pi_*\mathcal{O}_X$ is locally free.*

Thus, the local study of quotients of smooth varieties by finite groups reduces to the study of quotients of smooth varieties by small finite groups of automorphisms : Indeed, the theorem above implies that, locally around x, the group G_x/G_x^1 is a small group of automorphisms of the smooth variety X/G_x^1. It is also clear that, for local questions, by the Etale Slice Theorem (see (5)) one is reduced to study the case where X is a rational representation of G.

Theorem 6.1 *Let G be a finite abelian group acting on a smooth affine scheme X with quotient $\pi : X \longrightarrow Y$ and let k be an integer with $1 \leq k \leq \dim(X)$. The morphism $(d\pi^k)^G$ is surjective if and only if Y is smooth.*

Proof of 6.1 By the preceding remarks, we are reduced to the case where X is a rational representation of G as a small group of automorphism. So that the map π is unramified in codimension one.

We recall that, G being finite, we have $\Omega_{X,G} = \Omega_X$. Moreover by (5.1) we deduce that Y is smooth in codimension 2 and we can assume that $1 \leq k < \dim(X) - 1$. Thus we can assume that $\dim(X) > 2$ and purity of the branch locus implies that π is unramified in codimension 2.

From now on we proceed by induction on $\dim(X)$. Since G is abelian, X decomposes as a product of representation : $X = X' \times L$ with $2 \leq \dim(X') = \dim(X) - 1$. We have a diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

where the vertical maps are quotient by G and the horizontal ones are embeddings. This induces a commutative diagram :

$$\begin{tikzcd} (\Omega^k_X)^G \arrow[r] & (\Omega^k_{X'})^G \\ \Omega^k_Y \arrow[r] \arrow[u] & \Omega^k_{Y'} \arrow[u] \end{tikzcd}$$

where all the morphisms are surjective. Thus, by the induction hypothesis, Y' is smooth. Now, if G were not trivial, the origin being a fixed point, the map π' should have to be ramified and, by purity of the branch locus again, its ramification locus should have codimension one. But then π should be ramified in codimension 2. It is a contradiction. Thus, G is trivial and therefore Y is smooth. ■

A Regular and absolutely regular differentials

A.1 Regular differentials

Regular differentials together with duality theory have been studied by many authors but from different viewpoints. The main results that we need are found in the book of Kunz and Waldi ([KW88]), but we feel that the very general and explicit construction of regular differentials in this book (where the construction is local and relative from the beginning) asks a lot of the (lazy) reader, and therefore does not "specialize" easily to a convenient tool in the common case of schemes of finite type over a field.

Thus we choose the following path : We take the theory of the residual complex and fundamental class as exposed in the work of El Zein ([EZ78]) as a "black box" and rephrase, with a view toward Kunz and Waldi's theory of regular differentials, the results and constructions of El Zein. We do not intend to say anything new here and all the subsequent claims are implicitely proved in El Zein's article ([EZ78]). In fact, this approach was inspired to us by the work of Kersken ([Ker83b, Ker83a, Ker84]).

Construction

Let $\mathbf{k}$ be a field of characteristic 0. For any scheme X of finite type over $\mathbf{k}$, there exists a *residual complex* K_X ([Har66]). This is a complex of injective $\mathcal{O}_X$-modules concentrated in degree $[-\dim(X), 0]$, the image of which in the derived category is the *dualizing complex*.

Let $n = \dim(X)$. We denote by ω_X^n the module $\mathrm{H}^0(\mathrm{K}_X[-n])$. If X is smooth, K_X is the Cousin resolution of $\Omega_X^n[n]$. If $i : X \longrightarrow Y$ is an embedding of X into a smooth Y then $\mathrm{K}_X = i^!\mathrm{K}_Y = \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_X, \mathrm{K}_Y)$. If $\pi : X \longrightarrow Y$ is a finite surjective morphism then the complexes K_X and $\pi^!\mathrm{K}_Y$ are quasi-isomorphic and therefore $\omega_X^n \simeq \pi^!\omega_Y^n$. Moreover, the formation of the residual complex commutes with restriction to an open set. Thus, for a general X, ω_X^n has the S_2 property and coincides with Ω_X^n at the smooth points of X. Consequently, if X is normal then there is a natural isomorphism $\Omega_X^{n\,\vee\vee} \xrightarrow{\sim} \omega_X^n$.

The complex K_X is exact in degrees $\neq \dim(X)$ if and only if X is equidimensionnal and Cohen-Macaulay. In this case, the module ω_X^n is the *dualizing module* (usually denoted ω_X).

Now, following El Zein, let $\mathrm{K}_X^{*,\cdot} = \underline{\mathrm{Hom}}(\Omega_X, \mathrm{K}_X)$. It is a bigraded object, where the $*$ (resp. the $\cdot$) corresponds to degrees in Ω_X (resp. in K_X), concentrated in degrees $[-\infty, 0] \times [-\dim(X), 0]$. We now explain how one can put on $\mathrm{K}_X^{*,\cdot}$ a structure of complex of right differential graded Ω_X-modules concentrated in degree $[-\dim(X), 0]$.

The left Ω_X-module structure of Ω_X given by exterior product induces an obvious right Ω_X-module structure on $\mathrm{K}_X^{*,p} = \underline{\mathrm{Hom}}(\Omega_X, \mathrm{K}_X^p)$ and the differential δ of K_X induces an Ω_X-linear differential : $\delta' = \underline{\mathrm{Hom}}(\Omega_X, \delta)$.

The non-trivial point is the existence for all p of a differential endo-operator d' of order ≤ 1 and $*$-degree 1 on $\mathrm{K}_X^{*,p}$ satisfying the conditions

(i) $\delta'.d' = d'.\delta'$.

(ii) $d'(\phi.\alpha) = \phi.(d\alpha) + (-1)^q(\mathrm{d}'\phi).\alpha$, for $\alpha \in \Omega_X^q$ and $\phi \in \mathrm{K}_X^{*,p}$.

The construction of d' is explained in ([EZ78, 2.1.2]), the proof of (ii) follows from the lemma ([EZ78, 2.1.2, Lemme], be aware that there is a misprint in this paper : The logical section 2.1.2 is labelled 3.1.2) and the remarks following the proof of this lemma. Finally, (i) is a direct consequence of ([EZ78, 2.1, Proposition]) and ([EZ78, 2.1.2, Proposition]). We want to insist on the fact that, even in the smooth case, the operator d' is not the naive (and above all, meaningless) "$\underline{\mathrm{Hom}}(d, \mathrm{K}_X)$". We can now define the module of *regular differential forms* : $\omega_X = \mathrm{H}^{*,0}(\mathrm{K}_X^{*,\cdot}[-n,-n])$. Thus, ω_X is a right differential graded Ω_X-module and one has $\omega_X^k = \underline{\mathrm{Hom}}(\Omega_X^{n-k}, \omega_X^n)$.

When X is normal and equidimensional, the isomorphism ${\Omega_X^n}^{\vee\vee} \xrightarrow{\sim} \omega_X^n$ therefore induces an isomorphism ${\Omega_X}^{\vee\vee} \xrightarrow{\sim} \omega_X$. Thus, in this case, it is easily seen that this construction coincides with that of Kunz and Waldi ([KW88, 3.17, Theorem]). Note also that, when X is normal, ω_X is a reflexive module.

The fundamental class

The *fundamental class* is constructed and studied by El Zein in ([EZ78, 3.1, Théorème]). The fundamental class is defined as a global section C_X of $\mathrm{K}_X^{*,\cdot}$ (as a bigraded object) satisfying $d'\mathrm{C}_X = \delta'\mathrm{C}_X = 0$. When X is equidimensional of dimension n, the fundamental class is homogeneous of degree $(-n,-n)$. In general, the contribution to C_X of an m-dimensional irreducible component of X is homogeneous of degree $(-m,-m)$ (cf. the next section). Let X be an n-dimensionnal scheme. By this observation, since $\delta'\mathrm{C}_X = 0$, we have an induced cohomology class $\mathrm{c}_X \in \omega_X^0$. Then, right multiplication defines a morphism

$$\begin{array}{rcl} \Omega_X & \longrightarrow & \omega_X \\ \alpha & \longmapsto & \mathrm{c}_X.\alpha \end{array}$$

of differential graded Ω_X-modules, thanks to the relation $d'\mathrm{c}_X = 0$. We again denote by c_X this morphism and also call it the fundamental class morphism.

To be a little more explicit, $\mathrm{c}_X \in \mathrm{H}^0(X, \mathrm{K}_X^{*,\cdot}[-n,-n]) = \mathrm{Hom}(\Omega_X^n, \omega_X^n)$ and the fundamental class morphism in degree k is the composition

$$\Omega_X^k \longrightarrow \mathrm{Hom}(\Omega^{n-k}, \Omega_X^n) \longrightarrow \mathrm{Hom}(\Omega^{n-k}, \omega_X^n) \simeq \omega_X^k.$$

When X is normal and equidimensional, the morphism c_X can be identified with the natural morphism $\Omega_X \longrightarrow {\Omega_X}^{\vee\vee} \simeq \omega_X$.

We can now state the following fundamental theorem of Kunz and Waldi :

Theorem ([KW88, 5.22, p107]) *Let X be an equidimensional Cohen-Macaulay reduced scheme of finite type over* $\mathbf{k}$ *and let $n = \dim(X)$. Then the support of* $\mathrm{Coker}(\mathrm{c}_X)^n$ *is precisely the singular locus of X.*

The trace map for regular differentials

Let $f : X \longrightarrow Y$ be a proper morphism, then the trace morphism $\mathrm{Tr}f : f_*\mathrm{K}_X^{*,\cdot} \longrightarrow \mathrm{K}_Y^{*,\cdot}$ is obtained by the composition of the natural morphism $\Omega_Y \longrightarrow f_*\Omega_X$ with the trace morphism for residual complexes $f_*\mathrm{K}_X \longrightarrow \mathrm{K}_Y$. We thus have a well defined trace morphism $\mathrm{Tr}f : f_*\omega_X \longrightarrow \omega_Y$ vanishing if $\dim(X) \neq \dim(Y)$.

Assume that f is birational, i.e., that there exists a dense open subset $V \subset Y$ such that the induced morphism $f^{-1}(V) \longrightarrow V$ be an isomorphism. Then, by ([EZ78, 3.1, Théorème]) the trace morphism $\mathrm{Tr}f : f_*\mathrm{K}_X^{*,\cdot} \longrightarrow \mathrm{K}_Y^{*,\cdot}$ sends C_X to C_Y. Consequently, under these hypotheses we have a commutative diagram :

$$\begin{tikzcd} f_*\Omega_X \arrow[r, "\mathrm{c}_X"] & f_*\omega_X \arrow[d, "\mathrm{Tr}f"] \\ \Omega_Y \arrow[u] \arrow[r, "\mathrm{c}_Y"] & \omega_Y \end{tikzcd}$$

Let X be a scheme and $X_1, \ldots, X_k$ its irreducible components with their reduced structure and inclusions $j_i : X_i \subset X$. Then by construction ([EZ78, p37]) we have that $\mathrm{C}_X = \sum_i e_{X_i}(X)\mathrm{Tr}j_i(\mathrm{C}_{X_i})$, where $e_{X_i}(X) = \mathrm{length}(\mathcal{O}_{X,X_i})$, the multiplicity of X along X_i. Thus, we have $\mathrm{c}_X = \sum_i e_{X_i}(X)\mathrm{Tr}j_i(\mathrm{c}_{X_i})$.

Assume now that $f : X \longrightarrow Y$ is a finite dominant morphism between integral schemes then by ([EZ78, 3.1, Proposition 2]) we have that $\mathrm{Tr}f(\mathrm{C}_X) = \deg(f)\mathrm{C}_Y$. We therefore have a commutative diagram :

$$\begin{tikzcd} f_*\Omega_X \arrow[r, "\mathrm{c}_X"] & f_*\omega_X \arrow[d, "\mathrm{Tr}f"] \\ \Omega_Y \arrow[u] \arrow[r, "\deg(f)\mathrm{c}_Y"] & \omega_Y \end{tikzcd}$$

A.2 Absolutely regular differentials

Let X be a scheme and $f : \tilde{X} \longrightarrow X$ a desingularisation (if X is not reduced, by this, we mean a desingularisation of X_{red}). We recall that the $\mathcal{O}_X$-module $f_*\Omega_{\tilde{X}}$ is independent of the choice of f, we denote it by $\tilde{\Omega}_X$. It is usually called the module of *absolutely regular differentials*, or sometimes, when X is a normal variety, the module of *Zariski differentials*. By construction, we have natural morphisms

$$\Omega_X \longrightarrow \tilde{\Omega}_X \longrightarrow i_*\Omega_{X_{\mathrm{smth}}}$$

where i is the inclusion $X_{\mathrm{smth}} \subset X$. Therefore, when X is reduced, we have :

$$\Omega_X \longrightarrow \bar{\Omega}_X \subset \tilde{\Omega}_X \subset i_*\Omega_{X_{\mathrm{smth}}}.$$

In general, we also have a commutative diagram :

$$\begin{tikzcd} f_*\Omega_{\tilde{X}} \arrow[r, equal] & f_*\omega_{\tilde{X}} \arrow[d, "\mathrm{Tr}f"] \\ \Omega_X \arrow[u] \arrow[r, "\mathrm{c}_X"] & \omega_X \end{tikzcd}$$

and consequently, a sequence of morphisms

$$\Omega_X \longrightarrow \tilde{\Omega}_X \longrightarrow \omega_X.$$

Let $f : X \longrightarrow Y$ be a dominant morphism. Then we have a commutative diagram

$$\begin{tikzcd} \Omega_X \arrow[r] & \tilde{\Omega}_X \\ f^*\Omega_Y \arrow[u] \arrow[r] & f^*\tilde{\Omega}_Y \arrow[u] \end{tikzcd}$$

Assume moreover that the morphism f is proper and birational. Then we have a commutative diagram

$$\begin{tikzcd} f_*\Omega_X \arrow[r] & f_*\tilde{\Omega}_X \arrow[r] & f_*\omega_X \arrow[d, "\mathrm{Tr}f"] \\ \Omega_Y \arrow[u] \arrow[r] & \tilde{\Omega}_Y \arrow[u] \arrow[r] & \omega_Y \end{tikzcd}$$

where the rows are factorisations of the respective fundamental class morphisms. Note that–obviously–the middle vertical arrow is an isomorphism.

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